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Edge currents in non-commutative Chern-Simons theory from a new matrix model

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Abstract: This paper discusses the formulation of the non-commutative Chern-Simons (CS) theory where the spatial slice, an infinite strip, is a manifold with boundaries. As standard $*$-products are not correct for such manifolds, the standard non-commutative CS theory is not also appropriate here. Instead we formulate a new finite-dimensional matrix CS model as an approximation to the CS theory on the strip. A work which has points of contact with ours is due to Lizzi, Vitale and Zampini where the authors obtain a description for the fuzzy disc. The gauge fields in our approach are operators supported on a subspace of finite dimension $N+\eta$ of the Hilbert space of eigenstates of a simple harmonic oscillator with $N\,,\eta \in \mathbb{Z}^+$ and $N \neq 0$. This oscillator is associated with the underlying Moyal plane. The resultant matrix CS model has a fuzzy edge. It becomes the required sharp edge when N and $\eta \to \infty$ in a suitable sense. The non-commutative CS theory on the strip is defined by this limiting procedure. After performing the canonical constraint analysis of the matrix theory, we find that there are edge observables in the theory generating a Lie algebra with properties similar to that of a non-abelian Kac-Moody algebra. Our study shows that there are $(\eta+1)^2$ abelian charges(observables) given by the matrix elements $(\hat{\mathcal{A}}_i)_{N-1\,N-1}$ and $(\hat{\mathcal{A}}_i)_{nm}$ (where $n\,or\,m \geq N$) of the gauge fields, that obey certain standard canonical commutation relations. In addition, the theory contains three unique non-abelian charges, localized near the N^{th} level. We observe that all non-abelian edge observables except these three can be constructed from the $(\eta+1)^2$ abelian charges above. Using some of the results of this analysis we discuss in detail the limit where this matrix model approximates the CS theory on the infinite strip.

Keywords: Lattice Quantum Field Theory, Chern-Simons Theories, Non-Commutative Geometry, Matrix Models.

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1. Introduction

Recently, there has been much interest in formulating Chern-Simons theories on non-commutative manifolds [1]. In [2] the non-commutative CS theory has been used for the purpose of obtaining a new description of the Quantum Hall Effect (QHE). Polychronakos [3] and Morariu and Polychronakos [4] have proposed finite-dimensional matrix models of CS theory and used them to explain certain properties of the Fractional Quantum Hall Effect (FQHE).

In this paper we report on our work on the formulation of non-commutative CS theory where the spatial slice, an infinite strip, is a manifold with boundaries. It can serve to describe non-commutative QHE in such a strip.

There have been previous attempts to carry out such a formulation on a disc and on a half plane in the presence of spatial non-commutativity [5, 6]. The main obstacle faced in these attempts can be traced to the absence of a well-defined $\star$-product for these types of manifolds. We discuss this problem in the beginning of section 3 in some detail to better explain our motivation and the need for the matrix CS model we introduce afterwards. Recently, Pinzul and Stern [7] have studied the CS theory written on a non-commutative plane with a 'hole'. They have shown that in this case the algebra of observables is a nonlinear deformation of the w_∞ algebra. In another work [8], Lizzi, Vitale and Zampini have formulated a fuzzy disc, on which the non-commutative CS theory can be formulated. It has overlaps with our work here and has also points of contact with [5].

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Our model stems from taking gauge fields as operators supported on an $N+\eta$-dimensional subspace of the Hilbert space, where $N\,,\eta \in \mathbb{Z}^+$ and $N \neq 0$. It is spanned by the eigenstates of a simple harmonic oscillator associated to the underlying Moyal plane. The resultant matrix CS model has a fuzzy edge and it becomes the required sharp edge when N and $\eta \to \infty$ in a suitable sense. The non-commutative CS theory on the strip is defined by this limiting procedure. In this framework we perform the canonical constraint analysis for the finite-dimensional matrix CS model and find out that there are edge observables in the theory generating a Lie algebra with properties similar to that of a non-abelian Kac-Moody algebra. Our study shows that there are $(\eta+1)^2$ "abelian" charges (observables) given by the matrix elements $(\hat{\mathcal{A}}_i)_{N-1\,N-1}$ and $(\hat{\mathcal{A}}_i)_{nm}$ (where $n\,or\,m \geq N$) of the gauge fields, that obey certain standard canonical commutation relations. In addition, the theory contains three unique "non-abelian" charges, localized near the N^{th} level. We show that all "non-abelian" edge observables, except these three can be constructed from the $(\eta+1)^2$ "abelian" charges above. Using some of the results of this analysis we discuss in detail the limit where this matrix model approximates the CS theory on the infinite strip. Finally, we discuss the commutative limit of our model, where we also give a closed formula for the central charge of the Kac-Moody-like algebra of the non-commutative CS theory in terms of the diagonal coherent state matrix elements of operators and star products.

The organization of our paper is as follows: in section 2 we start with a brief review of the formulation of edge currents in a commutative CS theory on an infinite strip. Here we also demonstrate how one can formulate these edge effects while working on the full plane and encoding the boundary structure in the test functions. The next three sections 3, 4 and 5 systematically develop the ideas outlined in the last paragraph. We conclude with highlighting the main results of our work and their possible physical significance.

2. Edge currents revisited

Here we briefly review the derivation of Kac-Moody algebra obeyed by edge observables in an abelian Chern-Simons theory. Results summarized in this section are well-established and extensively discussed in [9, 10]. We consider the simplest case of an infinite strip (say in x_2 direction) in $\mathbb{R}^2 \otimes \mathbb{R}^1$, where $\mathbb{R}^1$ stands for time. This formulation closely follows the one given in [10] for a disc. Here we also describe another way in which we can work on the full plane and encode the boundary structure in the domains of the test functions. This will serve as a prototype approach we shall use for the finite-dimensional matrix CS model.

Let $x \equiv (x_1, x_2)$ denote the points of $\mathbb{R}^2$. The strip $\mathcal{T}$ is defined by

$$\mathcal{T} = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, -L \leq x_1 \leq L\}\,. \tag{2.1}$$

The usual action for the abelian Chern-Simons theory on $\mathcal{T}$ is

$$S = \frac{\kappa}{4\pi} \int_{\mathcal{T}\otimes\mathbb{R}^1} A \wedge dA \qquad A = A_\mu dx^\mu\,. \tag{2.2}$$

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The gauge fields satisfy the equal time Poisson Brackets (P.B.'s)

$$\{A_i(x_0,x_1,x_2)\,,A_j(x_0,x_1'\,,x_2')\} = \varepsilon_{ij}\frac{2\pi}{\kappa}\delta^2(\vec{x}-\vec{x}')\,, \tag{2.3}$$

where $i,j=1,2$ and $\varepsilon_{12}=-\varepsilon_{21}=1$.

From the variation of S under A_0 we have the Gauss law constraint, which can be written by introducing test functions Λ^0 as

$$g(\Lambda^0)=\frac{\kappa}{2\pi}\int_{\mathcal{T}}\Lambda^0(x)dA(x)\approx 0\,. \tag{2.4}$$

Differentiability of $g(\Lambda^0)$ imposes the condition $\Lambda^0|_{\partial\mathcal{T}}=0$. Using (2.3) and (2.4) it can be shown that $g(\Lambda^0)$ are first class constraints:

$$\{g(\Lambda^0_1)\,,g(\Lambda^0_2)\}\approx 0\,, \tag{2.5}$$

and generate the gauge transformations $A\to A-d\Lambda^0$.

The charges of this theory are

$$q(\Lambda)=\frac{\kappa}{2\pi}\int_{\mathcal{T}}d\Lambda\wedge A\,. \tag{2.6}$$

Since they are first class:

$$\{q(\Lambda)\,,g(\Lambda^0)\}\approx 0\,, \tag{2.7}$$

they constitute the observables of this theory. We note that Λ are test functions associated with $q(\Lambda)$'s and unlike Λ^0 they in general do not vanish on $\partial\mathcal{T}$.

Now note that for $\Lambda_1-\Lambda_2=\Lambda^0$,

$$q(\Lambda_1)-q(\Lambda_2)=-g(\Lambda_1-\Lambda_2)\approx 0\,. \tag{2.8}$$

This means that test functions which are equal on $\partial\mathcal{T}$ generate charges which are equal modulo constraints. Thus, $q(\Lambda)$ are indeed edge observables. They generate the transformations $A\to A-d\Lambda$ which do not necessarily vanish on $\partial\mathcal{T}$.

The P.B.'s of charges can be identified to be the $U(1)$ Kac-Moody algebra on $\partial\mathcal{T}$, since

$$\{q(\Lambda_1),q(\Lambda_2)\}=\frac{\kappa}{2\pi}\int_{\mathcal{T}}d\Lambda_1\wedge d\Lambda_2=\frac{\kappa}{2\pi}\int_{\partial\mathcal{T}}\Lambda_1\,d\Lambda_2\,. \tag{2.9}$$

Choosing the basis for the test functions on $\partial\mathcal{T}$ as

$$\Lambda_1|_{x_1=L}=e^{i\,k_1\,x_2}\,,\qquad \Lambda_2|_{x_1=L}=e^{i\,k_2\,x_2}\,,\qquad \Lambda_1|_{x_1=-L}=0\,,\qquad \Lambda_2|_{x_1=-L}=0\,, \tag{2.10}$$

(2.9) can be written as

$$\{q(\Lambda_1),q(\Lambda_2)\}=i\kappa\,k_2\delta(k_1+k_2)\,, \tag{2.11}$$

which is the usual form of the $U(1)$ Kac-Moody algebra.

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An equivalent formulation of the above can be given as follows. Consider the CS theory on $\mathbb{R}^2 \otimes \mathbb{R}^1$. However, now the spatial components of the gauge fields are supported in the region $|x_1| \leq L$ (i.e. in $\mathcal{T}$), whereas A_0 has support in $|x_1| < L$. Obviously, (2.3) holds now only if $x_1, x_1' \in \mathcal{T}$, otherwise its r.h.s. vanishes.

From the variation of S under A_0 we still have the Gauss law constraint

$$G(\Lambda^0) = \frac{\kappa}{2\pi} \int_{\mathbb{R}^2} \Lambda^0 dA \approx 0\,, \tag{2.12}$$

where now the integration is over $\mathbb{R}^2$. The condition that Λ^0's are supported in $|x_1| < L$ follows from that of δA_0 which is of course the same as that of A_0. Thus we have

$$\Lambda^0 = 0 \qquad \text{for} \qquad |x_1| \geq L\,. \tag{2.13}$$

The results of the canonical analysis (given in (2.5)–(2.7)) go through where now integrals are over all $\mathbb{R}^2$ and Λ^0 is as given in (2.13). This establishes $G(\Lambda^0)$'s as first class constraints and

$$Q(\Lambda) = \frac{\kappa}{2\pi} \int_{\mathbb{R}^2} d\Lambda \wedge A \tag{2.14}$$

as the observables. Clearly, Λ can be supported on all of $\mathbb{R}^2$, but if it is supported only on $\mathbb{R}^2 \setminus \mathcal{T}$ (i.e. $|x_1| > L$) then we immediately see from (2.14) that $Q(\Lambda) \equiv 0$.

For $\Lambda_1 - \Lambda_2 = \Lambda^0$ we have

$$Q(\Lambda_1) - Q(\Lambda_2) = -G(\Lambda_1 - \Lambda_2) \approx 0\,. \tag{2.15}$$

It follows from this and the remark after (2.14) that $Q(\Lambda)$ are observables localized at $x_1 = \pm L$. The P.B.'s of charges gives the Kac-Moody algebra:

$$\{Q(\Lambda_1), Q(\Lambda_2)\} = \frac{\kappa}{2\pi} \int_{\mathbb{R}^2} d\Lambda_1 \wedge d\Lambda_2\,. \tag{2.16}$$

A suitable choice for Λ's to compute this algebra at $x_1 = L$ is

$$\Lambda_i = \theta(x_1 - L)\, e^{i\, k_i\, x_2}\,, \qquad (i = 1, 2)\,, \tag{2.17}$$

where $\theta(x_1 - L)$ is the step function centered at $x_1 = L$ with $\theta(0) = 1$. Then we have from (2.16) that

$$\{Q(\Lambda_1), Q(\Lambda_2)\} = i\kappa\, k_2 \delta(k_1 + k_2)\,, \tag{2.18}$$

which is the same as (2.11). In order to get the algebra of observables at $x_1 = -L$ one has to replace (2.17) by

$$\Lambda_i = (1 - \theta(x_1 + L))\; e^{i\, k_i\, x_2}\,. \tag{2.19}$$

This way of treating edge properties is not completely new and has examples in planar systems of condensed matter physics [11]. We now turn our attention to the treatment of the non-commutative case.

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3. Non-commutative Chern-Simons theory on the infinite strip

3.1 Remarks on the non-commutative CS theory on a manifold with boundaries

As pointed out by [5, 6] the canonical formulation of Chern-Simons theory on a non-commutative manifold with boundaries (such as a disc, an infinite strip etc.) presents serious difficulties. In fact the problem underlying such a formulation is that the standard $\star$-product on the Moyal plane is no longer valid on manifolds with boundaries. To make this point clear and supply enough mathematical basis for this assertion let us consider first the Chern-Simons action on the Moyal plane with no boundaries:

$$S_{NCCS} = -\frac{\kappa}{4\pi}\int dx_0\, d^2x\, \varepsilon_{\mu\nu\lambda}\left(A_\mu \star_M \partial_\nu A_\lambda + \frac{2}{3} A_\mu \star_M A_\nu \star_M A_\lambda\right). \tag{3.1}$$

Here $(\mu,\nu,\lambda = 0,1,2)$, x_0 is time x_1 and x_2 are coordinates on the Moyal plane and $\varepsilon_{123} = 1$.

The Moyal algebra is characterized by the $\star$-product

$$f \star_M g(x_1,x_2) = f(x_1,x_2) e^{\frac{i\theta}{2}(\overleftarrow{\partial}_{x_1}\overrightarrow{\partial}_{x_2} - \overleftarrow{\partial}_{x_2}\overrightarrow{\partial}_{x_1})} g(x_1,x_2)\,. \tag{3.2}$$

Defining the $\star$-commutator of f and g by

$$[f,g]_{\star_M} = f \star_M g - g \star_M f\,, \tag{3.3}$$

the spatial non-commutativity can be expressed as

$$[x_1\,, x_2]_{\star_M} = i\theta\,, \tag{3.4}$$

θ being the non-commutativity parameter.

Naively, one may consider S_{NCCS} on a manifold with boundaries, say the infinite strip $\mathcal{T}$, and write the Gauss law constraint as

$$g(\Lambda^0) = -\frac{\kappa}{2\pi}\int_{\mathcal{T}} d^2\, x\, \varepsilon_{ij}\, \Lambda^0\, (\partial_i\, A_j + A_i \star_M A_j) \approx 0\,, \tag{3.5}$$

where $(i,j = 1,2)$ and $\Lambda^0|_{\partial\mathcal{T}} = 0$. However, this expression as well as the S_{NCCS} when written on the strip $\mathcal{T}$ is not well defined. This is because $\star_M$ product does not exist on $\mathcal{T}$. To prove this fact we note that the formula for the $\star_M$ product in (3.2) contains the exponential of the differential operator $-i\partial_{x_1}$. With the usual definition of its domain $-i\partial_{x_1}$ generates translations so that $e^{i(-ic\partial_{x_1})}$ translates functions of x_1 by c : $(e^{i(-ic\partial_{x_1})})\psi(x_1) = \psi(x_1+c)$. Consequently, if ψ has support $[-L,L]$, $e^{i(-ic\partial_{x_1})}\psi$ does not, and $\star_M$ is not defined on functions supported in $[-L,L]$ [12].

Alternatively, to circumvent the impossible task of obtaining a well-defined and useful $\star$-product on these types of manifolds we propose a finite-dimensional matrix model where the edges are fuzzy. It becomes the CS theory on a non-commutative infinite strip in the limit where the size of the matrices approaches infinity. In this respect, we emphasize that our approach is completely different from the previous attempts in the literature as the following section illustrates.

3.2 The matrix model

We now describe our matrix model. Since we will be working in the operator formalism in this subsection, we discriminate the operators from the elements of the corresponding Moyal $\star$ algebra of functions by putting a hat symbol on the elements of the former. Thus the Moyal plane is described by operators $\hat{x}_i\,(i=1,2)$ with the relation

$$[\hat{x}_1, \hat{x}_2] = i\theta \,. \tag{3.6}$$

To supply the mathematical basis for our arguments we think of a simple harmonic oscillator in the x_1-direction, described by the hamiltonian

$$\hat{H} = \frac{\hat{x}_2^2}{2m} + \frac{1}{2}k\hat{x}_1^2 \,, \tag{3.7}$$

and the oscillation frequency $\omega = \sqrt{\frac{k}{m}}$. Now consider the Hilbert space $\mathcal{H}$ spanned by the eigenstates of this hamiltonian. In the matrix CS model we will construct $\mathcal{H}$ will be associated to the underlying Moyal plane, and its finite-dimensional subspaces will serve as the carrier spaces of the operators that we are going to introduce in our matrix model. More precisely, we will define these operators in terms of their action on the elements of a finite-dimensional subspace of $\mathcal{H}$ and its orthogonal complement.

The number of energy eigenstates of this hamiltonian below the energy $E = \frac{1}{2}kL^2$ with L being the maximum classical amplitude (given by the location of the edges) is finite and given by

$$M = \left[\frac{kL^2 + \theta\omega}{2\theta\omega}\right] , \tag{3.8}$$

where $[\frac{kL^2+\theta\omega}{2\theta\omega}]$ is the largest integer smaller than $\frac{kL^2+\theta\omega}{2\theta\omega}$. These M states span a subspace $\mathcal{H}_M$ of the harmonic oscillator Hilbert space $\mathcal{H}$ and they can be taken as an orthonormal basis in $\mathcal{H}_M$.[1] From (3.8) it is easy to see that keeping both oscillation frequency ω and the maximum oscillation amplitude L fixed while increasing k (i.e. steepening the potential) results in larger number of states with energy $E \leq \frac{1}{2}kL^2$. Hereafter we keep ω fixed unless otherwise stated. Let us consider a level K with $K \leq M$. Then the large $K \approx M$ levels with energy $E \approx \frac{1}{2}kL^2$ get localized near the edges $x_1 = \pm L$, since their probability amplitudes are maximum there. On the contrary, those levels with $E \ll \frac{1}{2}kL^2$ are localized well inside $-L \leq x_1 \leq L$ for large M. We also note that all levels decay exponentially outside $|x_1| \leq L$. Linear operators on $\mathcal{H}_M$ which are zero on the orthogonal complement $\mathcal{H}_M^\perp$ behave also in a similar fashion. Later we will show that diagonal coherent state matrix elements of operators, say like $|K><K|$ with $K \approx M$ will peak near $\pm L$ for all $|x_2| < M$ for large M, while those with K much less than M acquire maxima within $|x_1| < L$. The characteristic width of the peaks at $\pm L$, which gets narrower as M gets larger, gives us a natural scale to which we will relate the thickness $\Delta\ell$ of the "fuzzy edge" of our matrix model.

[1] To avoid any confusion that our notation might create later on we note that the Fock basis we are using for $\mathcal{H}$ is the usual one where states are labeled starting with quantum number 0, therefore the top state in $\mathcal{H}_M$ has the quantum number $M-1$.

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For the finite-dimensional matrix CS theory we have in mind, we treat spatial components of the gauge fields as operators supported on this M-dimensional subspace $\mathcal{H}_M$ of $\mathcal{H}$. This and the physical picture described above for the underlying Moyal plane imply two immediate consequences:

- It is those states contained within the thickness $\Delta\ell$ of the edges that will be responsible for the edge observables in our finite-dimensional matrix CS model;
- The $M \to \infty$ limit of this matrix CS theory can be taken to define the non-commutative CS theory on the infinite strip with boundaries at $x_1 = \pm L$.

After writing the matrix CS model and performing the constraint analysis for it, we will find these observables, and we will have the necessary information to explain and elaborate on the properties of the large M limit and how it is realized.

We now split M as $M = N + \eta$ where $N\,, \eta \in \mathbb{Z}^+$ and $N \neq 0$, and use this separation of the total dimension to define the domains and the ranges of our operators. In our formalism the gauge fields $\hat{A}_\mu\,, (\mu = 0, i$ where $i = 1, 2)$ are anti-hermitean operators that we choose to take in the way given below:

$$\begin{aligned} &\hat{A}_i \mathcal{H}_{N+\eta} \subseteq \mathcal{H}_{N+\eta}\,, \qquad && \hat{A}_i \mathcal{H}^{\perp}_{N+\eta} = \{0\}\,, \\ &\hat{A}_0 \mathcal{H}_{N-1} \subseteq \mathcal{H}_{N-1}\,, \qquad && \hat{A}_0 \mathcal{H}^{\perp}_{N-1} = \{0\}\,. \end{aligned} \tag{3.9}$$

Here for any $K \in \mathbb{Z}^+$,$\mathcal{H}^{\perp}_K$ denotes the orthogonal complement of $\mathcal{H}_K$. The fact that $\hat{A}_0$ is nonzero only on $\mathcal{H}_{N-1}$ and not on $\mathcal{H}_{N+\eta}$ should be noted. The reason behind this condition will be explained later in connection with the canonical analysis of the model.

If $|n>$ denotes the n^{th} normalized energy level and $\hat{P}_{nm}$ is the operator given by $\hat{P}_{nm} = |n><m|$, then we can write,

$$\hat{A}_i = \sum_{n,m=0}^{(N-1+\eta)} i(\hat{\mathcal{A}}_i)_{nm}\,\hat{P}_{nm}\,, \quad (\hat{\mathcal{A}}_i)_{nm} = 0 \quad \text{for} \quad n \quad \text{or} \quad m > N-1+\eta\,. \tag{3.10}$$

For $\hat{A}_0$, the same equation is valid if we replace $N-1+\eta$ with $N-2$. It also follows from $\hat{A}^{\dagger}_\mu = -\hat{A}_\mu$ that

$$(\hat{\mathcal{A}}_\mu)^*_{nm} = -(\hat{\mathcal{A}}_\mu)_{mn}\,. \tag{3.11}$$

There is another way to express (3.9). For this let us introduce the orthogonal projector $\hat{\mathbf{1}}_K$ by

$$\begin{aligned} &\hat{\mathbf{1}}_K = \sum_{n\,,m=0}^{K-1} (\hat{\mathbf{1}}_K)_{nm}\hat{P}_{nm} := \sum_{n\,,m=0}^{K-1} (\delta)_{nm}\hat{P}_{nm} = \sum_{n=0}^{K-1} \hat{P}_{nn}\,, \\ &(\hat{\mathbf{1}}_K)_{nm} = (\delta)_{nm} = 0 \quad \text{for} \quad n \quad \text{or} \quad m > K-1\,. \end{aligned} \tag{3.12}$$

Then (3.9) is equivalent to

$$\hat{A}_i = \hat{\mathbf{1}}_{N+\eta}\,\hat{A}_i\,\hat{\mathbf{1}}_{N+\eta}\,, \qquad \hat{A}_0 = \hat{\mathbf{1}}_{N-1}\,\hat{A}_0\,\hat{\mathbf{1}}_{N-1}\,. \tag{3.13}$$

The Chern-Simons lagrangian for the model reads (up to a total space derivative)

$$L_{\rm NCCS} = -\frac{\kappa\theta}{2}\varepsilon_{ij}\,{\rm Tr}\,\big(-\hat{A}_i\dot{\hat{A}}_j + 2\hat{A}_0(\partial_i\hat{A}_j + \hat{A}_i\hat{A}_j)\big)\,, \tag{3.14}$$

where $\dot{\hat{A}}_j = \partial_0\hat{A}_j$, trace is over the Hilbert space $\mathcal{H}$, and derivations are given by

$$\partial_i(.) = \frac{i}{\theta}\,\varepsilon_{ij}[\hat{x}_j\,,(.)]\,. \tag{3.15}$$

Several remarks about properties of $L_{\rm NCCS}$ are in order. First, $L_{\rm NCCS}$ changes by total derivatives under infinitesimal gauge transformations of the form

$$\hat{A}_\mu \to \hat{A}_\mu + (\partial_\mu\hat{\lambda} + i[\hat{A}_\mu\,,\hat{\lambda}])\,, \tag{3.16}$$

where $\hat{\lambda}$ is a matrix with infinitesimal elements. Next, as in Chern-Simons theory on a commutative manifold, the conjugate momenta $\hat{\Pi}_0$ to $\hat{A}_0$ are weakly equal to zero and first class, thus $\hat{A}_0$ is not an observable and can be eliminated from the rest of our discussion.

The equal time P.B.s of $(\hat{\mathcal{A}}_i)_{nm}$ can be written as

$$\{(\hat{\mathcal{A}}_i)_{nm}\,,(\hat{\mathcal{A}}_j)_{rs}\} = \frac{1}{\kappa\theta}\varepsilon_{ij}(\hat{\mathbf{1}}_{N+\eta})_{ns}(\hat{\mathbf{1}}_{N+\eta})_{mr} = \frac{1}{\kappa\theta}\varepsilon_{ij}\delta_{ns}\delta_{mr}\,, \qquad n,m,r,s\in[0,N-1+\eta]\,, \tag{3.17}$$

which in the operator formalism is the statement that $(\hat{\mathcal{A}}_1)_{nm}$ and $(\hat{\mathcal{A}}_2)_{rs}$ are canonically conjugate. In terms of the operators $\hat{A}_i$ in (3.10), this is

$$\{\hat{A}_i\,,\hat{A}_j\} = -\frac{1}{\kappa\theta}(N+\eta)\,\varepsilon_{ij}\sum_{n=0}^{N-1+\eta}\hat{P}_{nn} = -\frac{1}{\kappa\theta}(N+\eta)\varepsilon_{ij}\,\hat{\mathbf{1}}_{N+\eta}\,. \tag{3.18}$$

We now turn our attention to the canonical constraint analysis of this model.

3.3 Canonical analysis of the matrix model

From the variation of $L_{\rm NCCS}$ with respect to $\hat{A}_0$, we have the Gauss law constraint

$$-\kappa\,\theta\,\varepsilon_{ij}\,{\rm Tr}\,(\delta\hat{A}_0(\partial_i\hat{A}_j + \hat{A}_i\hat{A}_j)) \approx 0\,. \tag{3.19}$$

As $\delta\hat{A}_0$ is not zero only in $\mathcal{H}_{N-1}$, we find

$$g(\hat{\Lambda}^0) = \kappa\,\theta\,\varepsilon_{ij}\,{\rm Tr}\,(\hat{\Lambda}^0(\partial_i\hat{A}_j + \hat{A}_i\hat{A}_j)) \approx 0\,, \tag{3.20}$$

where $\hat{\Lambda}^0$ is of the same form as $\delta\hat{A}_0$:

$$\hat{\Lambda}^0\mathcal{H}_{N-1} \subseteq \mathcal{H}_{N-1}\,, \qquad \hat{\Lambda}^0\mathcal{H}^{\perp}_{N-1} = \{0\} \tag{3.21}$$

and where we have changed the sign of (3.20) compared to (3.19) for future convenience. In terms of the orthogonal projectors introduced in (3.12),

$$\hat{\Lambda}^0 = \hat{\mathbf{1}}_{N-1}\,\hat{\Lambda}^0\,\hat{\mathbf{1}}_{N-1}\,. \tag{3.22}$$

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In the basis spanned by $\{i\hat{P}_{nm}\}$ we have

$$\hat{\Lambda}^0 = \sum_{n,m=0}^{N-2} i(\hat{\Lambda}^0)_{nm}\,\hat{P}_{nm}\,, \qquad (\hat{\Lambda}^0)_{nm} = 0 \quad \text{for} \quad n \quad \text{or} \quad m > N-2\,. \tag{3.23}$$

Anti-hermiticity requires that $(\hat{\Lambda}^0)^*_{nm} = -(\hat{\Lambda}^0)_{mn}$.

Equation (3.20) is the statement of Gauss law in our matrix model and $\hat{\Lambda}^0$'s are non-commutative analogues of the test functions of the commutative CS theory.

"Integrating" by parts $g(\hat{\Lambda}^0)$ can be written as

$$g(\hat{\Lambda}^0) = \kappa\,\theta\,\varepsilon_{ij}\;\mathrm{Tr}\;(\partial_i(\hat{\Lambda}^0\hat{A}_j) - \partial_i\hat{\Lambda}^0\hat{A}_j + \hat{\Lambda}^0\hat{A}_i\hat{A}_j) \approx 0\,. \tag{3.24}$$

Here the first term is the trace of a "total derivative" on a finite-dimensional Hilbert space and it vanishes. Note that $\partial_i(\hat{\Lambda}^0\hat{A}_j)$ is identically zero in $\{\mathcal{H}^{\perp}_{N+\eta+1}\}$. Hence, in our matrix model we can write the Gauss law as

$$g(\hat{\Lambda}^0) = \kappa\,\theta\,\varepsilon_{ij}\;\mathrm{Tr}\;(\,-\,\partial_i\hat{\Lambda}^0\hat{A}_j + \hat{\Lambda}^0\hat{A}_i\hat{A}_j) \approx 0\,. \tag{3.25}$$

The conditions on $\hat{\Lambda}^0$ translate to those on $\partial_i\,\hat{\Lambda}^0$ as

$$(\partial_i\,\hat{\Lambda}^0)\mathcal{H}_N \subseteq \mathcal{H}_N\,, \qquad (\partial_i\,\hat{\Lambda}^0)\mathcal{H}^{\perp}_N = \{0\}\,, \tag{3.26}$$

or

$$(\partial_i\hat{\Lambda}^0) = \mathbf{\hat{1}}_N\,(\partial_i\hat{\Lambda}_0)\,\mathbf{\hat{1}}_N\,. \tag{3.27}$$

Consider now the quantity

$$q(\hat{\Sigma}) = \kappa\,\theta\,\varepsilon_{ij}\;\mathrm{Tr}(-\partial_i\widehat{\Sigma}\hat{A}_j\;+\widehat{\Sigma}\hat{A}_i\hat{A}_j)\,, \tag{3.28}$$

for an arbitrary operator $\widehat{\Sigma}$. A straightforward calculation shows that

$$\{q(\widehat{\Sigma}_1)\,,q(\widehat{\Sigma}_2)\} = -q([\widehat{\Sigma}_1\,,\widehat{\Sigma}_2]) - \kappa\,\theta\,\varepsilon_{ij}\;\mathrm{Tr}\;\mathbf{\hat{1}}_{N+\eta}\,(\partial_i\widehat{\Sigma}_1)\,\mathbf{\hat{1}}_{N+\eta}\,(\partial_j\widehat{\Sigma}_2)\,. \tag{3.29}$$

For $\widehat{\Sigma}_i = \hat{\Lambda}^0_i$ $(i = 1, 2)$ we have

$$q(\hat{\Lambda}^0_i) = g(\hat{\Lambda}^0_i)\,. \tag{3.30}$$

Also with this substitution the central term in (3.29) becomes

$$\begin{aligned} -\kappa\,\theta\,\varepsilon_{ij}\;\mathrm{Tr}\;\mathbf{\hat{1}}_{N+\eta}\,(\partial_i\hat{\Lambda}^0_1)\,\mathbf{\hat{1}}_{N+\eta}\,(\partial_j\hat{\Lambda}^0_2) &= -\kappa\,\theta\,\varepsilon_{ij}\;\mathrm{Tr}(\partial_i\hat{\Lambda}^0_1)\,(\partial_j\hat{\Lambda}^0_2) \\ &= -\kappa\,\theta\,\varepsilon_{ij}\;\mathrm{Tr}\;\partial_i(\hat{\Lambda}^0_1\,\partial_j\hat{\Lambda}^0_2) = 0\,, \end{aligned} \tag{3.31}$$

where we have once more made use of the fact that the trace of a total derivative term vanishes on a finite-dimensional Hilbert space. Thus for the P.B.'s of $g(\hat{\Lambda}^0)$ we find

$$\{g(\hat{\Lambda}^0_1), g(\hat{\Lambda}^0_2)\} = -g([\hat{\Lambda}^0_1\,,\hat{\Lambda}^0_2]) \approx 0\,, \tag{3.32}$$

assuring that $g(\hat{\Lambda}^0)$ are first class constraints.

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Consider now $\widehat{\Sigma} = \hat{\Lambda} = \hat{\Lambda}' + \hat{\Lambda}^0$ for some $\hat{\Lambda}^0$ fulfilling (3.21) or equivalently (3.22) and $\hat{\Lambda}'$ such that

$$\begin{aligned} \hat{\Lambda}'\mathcal{H}_{N-1} &= 0\,, & \hat{\Lambda}'\mathcal{H}^{\perp}_{N-1} &\subseteq \mathcal{H}^{\perp}_{N-1}\,, \\ (\partial_i\hat{\Lambda}')\mathcal{H}_{N-2} &= 0\,, & (\partial_i\hat{\Lambda}')\mathcal{H}^{\perp}_{N-2} &\subseteq \mathcal{H}^{\perp}_{N-2}\,, \end{aligned} \tag{3.33}$$

or

$$0 = \hat{\mathbf{1}}_{N-1}\,\hat{\Lambda}'\,\hat{\mathbf{1}}_{N-1}\,. \tag{3.34}$$

In the basis spanned by $\{i\hat{P}_{nm}\}$ we have

$$\hat{\Lambda}' = \sum_{n,m=N-1}^{\infty} i\hat{\Lambda}'_{nm}\,\hat{P}_{nm}\,, \qquad \hat{\Lambda}'_{nm} = 0 \quad \text{for} \quad n \quad \text{or} \quad m < N-1\,. \tag{3.35}$$

Anti-hermiticity gives $(\hat{\Lambda}')^*_{nm} = -(\hat{\Lambda}')_{mn}$.

Now note that

$$q(\hat{\Lambda}) = q(\hat{\Lambda}' + \hat{\Lambda}^0) = q(\hat{\Lambda}') + g(\hat{\Lambda}^0) \approx q(\hat{\Lambda}')\,. \tag{3.36}$$

Next, from (3.21) and (3.33) we observe that

$$[\hat{\Lambda}'\,, \hat{\Lambda}^0] = 0\,. \tag{3.37}$$

Hence, we have

$$[\hat{\Lambda}\,, \hat{\Lambda}^0_2] = [\hat{\Lambda}' + \hat{\Lambda}^0_1\,, \hat{\Lambda}^0_2] = [\hat{\Lambda}^0_1\,, \hat{\Lambda}^0_2] \tag{3.38}$$

which is of the form fulfilling (3.21).

It now follows at once that the P.B.s of $q(\hat{\Lambda})$ with the Gauss law is

$$\begin{aligned} \{q(\hat{\Lambda})\,, q(\hat{\Lambda}^0_2)\} &= \{q(\hat{\Lambda})\,, g(\hat{\Lambda}^0_2)\} \\ &= -q([\hat{\Lambda}\,, \hat{\Lambda}^0_2]) - \kappa\,\theta\,\varepsilon_{ij}\ \mathrm{Tr}\ \hat{\mathbf{1}}_{N+\eta}\,(\partial_i\hat{\Lambda})\,\hat{\mathbf{1}}_{N+\eta}\,(\partial_j\hat{\Lambda}^0_2) \\ &= -q([\hat{\Lambda}'\,, \hat{\Lambda}^0_2]) - q([\hat{\Lambda}^0_1\,, \hat{\Lambda}^0_2]) - \kappa\,\theta\,\varepsilon_{ij}\ \mathrm{Tr}\ \hat{\mathbf{1}}_{N+\eta}\,(\partial_i\hat{\Lambda}')\,\hat{\mathbf{1}}_{N+\eta}\,(\partial_j\hat{\Lambda}^0_2) \\ &= -g([\hat{\Lambda}^0_1\,, \hat{\Lambda}^0_2]) \approx 0\,, \end{aligned} \tag{3.39}$$

where the "central term" vanishes. This is because $\partial_i\hat{\Lambda}'$ is projected to $\mathcal{H}_{N+\eta}$, and due to antisymmetry of the indices it becomes

$$-\kappa\,\theta\,\varepsilon_{ij}\ \mathrm{Tr}\ \partial_j(\hat{\mathbf{1}}_{N+\eta}\,\partial_i\hat{\Lambda}'\,\hat{\mathbf{1}}_{N+\eta}\hat{\Lambda}^0_2) \tag{3.40}$$

which is zero on a finite-dimensional Hilbert space.

In fact one can take the vanishing of the central term as a requirement on $\hat{\Lambda}^0$'s and find out that the maximally large subspace of $\mathcal{H}$ where $\hat{\Lambda}^0$ is supported and for which (3.37) holds is $\mathcal{H}_{N-1}$. This explains how we arrive at the particular form of the operators for $\hat{A}_0$ and $\hat{\Lambda}^0$ given in (3.9) and (3.21) respectively.

As a consequence of (3.39), $q(\hat{\Lambda})$'s with $\hat{\Lambda} = \hat{\Lambda}' + \hat{\Lambda}^0$ are first class, and they constitute a set of non-abelian observables of our matrix model.

Furthermore, for

$$\hat{\Lambda}_1 = \hat{\Lambda}' + \hat{\Lambda}^0_1\,, \qquad \hat{\Lambda}_2 = \hat{\Lambda}' + \hat{\Lambda}^0_2\,, \tag{3.41}$$

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we have

$$q(\hat{\Lambda}_1) - q(\hat{\Lambda}_2) = g(\hat{\Lambda}_1 - \hat{\Lambda}_2) \approx 0\,, \tag{3.42}$$

implying that the actions of $q(\hat{\Lambda}_1)$ and $q(\hat{\Lambda}_2)$ on physical states give the same result. Finally, we compute the P.B.s of $q(\Lambda)$'s and find that

$$\{q(\hat{\Lambda}_1)\,, q(\hat{\Lambda}_2)\} = -q([\hat{\Lambda}_1\,, \hat{\Lambda}_2]) - \kappa\,\theta\,\varepsilon_{ij}\ \mathrm{Tr}\ \hat{\mathbf{1}}_{N+\eta}\,(\partial_i \hat{\Lambda}_1)\,\hat{\mathbf{1}}_{N+\eta}\,(\partial_j \hat{\Lambda}_2)\,. \tag{3.43}$$

In order to discuss implication of these results we need one more ingredient, this is contributed by the P.B.

$$\{(\hat{\mathcal{A}}_l)_{nm}\,, g(\hat{\Lambda}^0)\} = (\partial_l \hat{\Lambda}^0)_{nm} - i[\hat{\Lambda}^0\,, \hat{\mathcal{A}}_l]_{nm} \approx 0\,, \tag{3.44}$$

for $n\,or\,m \geq N$ and for $n = m = N-1$. Thus $(\hat{\mathcal{A}}_l)_{nm}$ for $n\,or\,m \geq N$ and $(\hat{\mathcal{A}}_i)_{N-1\,N-1}$ are observables of our matrix theory. The algebra of these observables is standard and given by (3.17) together with the same restrictions on indices r and s as on n and m. It is easy to see that for fixed N and η there are $(\eta+1)^2$ of these observables. For instance, for $\eta = 0$ we have $(\hat{\mathcal{A}}_i)_{N-1\,N-1}$ as an observable, for $\eta = 1$, we have $(\hat{\mathcal{A}}_i)_{N-1\,N-1}$, $(\hat{\mathcal{A}}_i)_{N\,N}\,, (\hat{\mathcal{A}}_i)_{N\,N-1}\,, (\hat{\mathcal{A}}_i)_{N-1\,N}$ as observables.

From (3.43) we see that the observables $q(\hat{\Lambda})$ generate a finite-dimensional Lie algebra with properties similar to a non-abelian Kac-Moody algebra. The differences are that this Lie algebra is finite-dimensional and its central charge is modified, due to the appearance of the projectors $\hat{\mathbf{1}}_{N+\eta}$ in the expression.

Independently of the value of η, $q(\hat{\Lambda})$ is nonzero for nonzero entries of $\hat{\Lambda}_{N-1\,N-1}$, $\hat{\Lambda}_{N-1\,N}$ and $\hat{\Lambda}_{N\,N-1}$ in a given $\hat{\Lambda}$. Thus we have three unique non-abelian Kac-Moody-like observables. For fixed N and η the rest of such non-abelian observables are $(\eta+2)^2-1-3 = \eta(\eta+4)$ in number and they can be constructed from the $(\eta+1)^2$ observables $(\hat{\mathcal{A}}_i)_{nm}$ given above.

We now investigate the limit $M \to \infty$.

4. The large M limit

First we introduce the coherent state $|z\rangle$ by

$$|z\rangle = e^{-\frac{1}{2\theta}|z|^2} \sum_{r=0}^{\infty} \frac{z^r}{\sqrt{\theta^r r!}}\,|r\rangle\,, \tag{4.1}$$

where $\hat{a}|z\rangle = z|z\rangle$, $\langle z|z\rangle = 1$ and $[\hat{a}\,, \hat{a}^\dagger] = \theta$.

The diagonal coherent state element of the operator $\hat{P}_{M-1\,M-1} = |M-1\rangle\langle M-1|$ reads

$$\mathcal{P}_{M-1}(z,\bar{z}) = \frac{1}{\pi\theta}\langle z|M-1\rangle\langle M-1|z\rangle = e^{\frac{-|z|^2}{\theta}}\frac{|z|^{2(M-1)}}{\pi\theta^M (M-1)!}\,, \tag{4.2}$$

where we have included the normalization factor $1/\pi\theta$ so that $\int d^2z \mathcal{P}(z,\bar{z}) = 1$. From the definitions

$$\hat{x}_1 = \frac{L}{2\sqrt{(M-1/2)\,\theta}}(\hat{a} + \hat{a}^\dagger)\,, \qquad \hat{x}_2 = -\frac{i\sqrt{(M-1/2)\,\theta}}{L}(\hat{a} - \hat{a}^\dagger)\,, \tag{4.3}$$

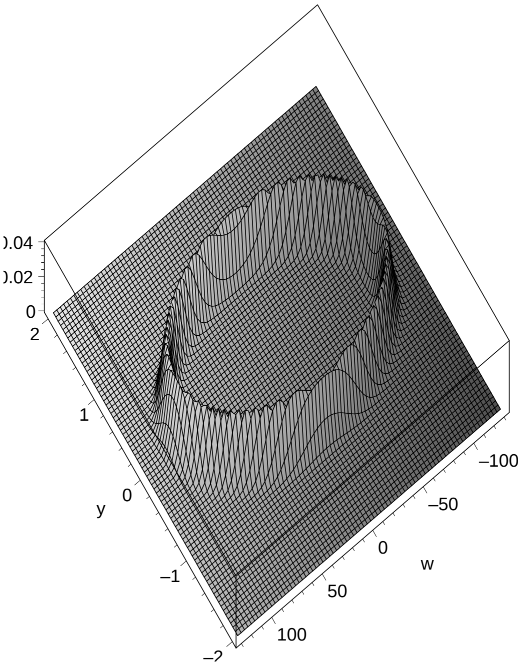

Figure 1: Sample plot of $\mathcal{Q}_{M-1}(x_1\,,x_2)$ for $M = 100$. Here the axes are labeled by $y := \frac{x_1}{L}$ and $\omega := \frac{L}{2\theta}x_2$. The geometry approximates an infinite strip along w axis with peaks localized at $y = \pm 1$ as $M \to \infty$.

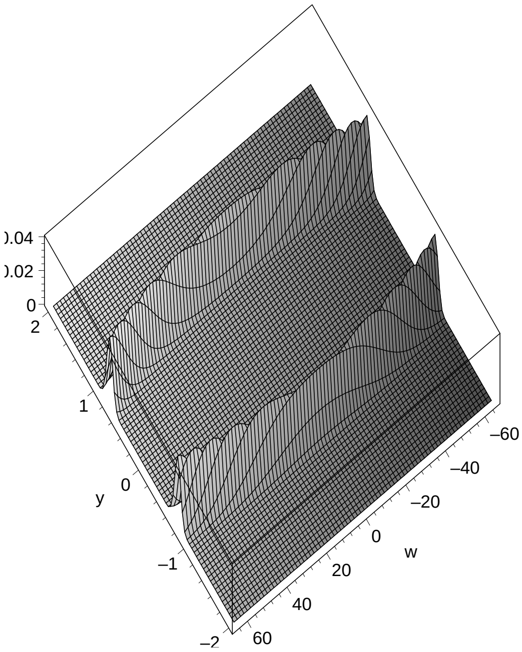

Figure 2: Blow up of $\mathcal{Q}_{M-1}(x_1\,,x_2)$ for $M = 100$ and $|\omega| \leq 65$.

we can deduce the relation between $z = (z_1\,,z_2)$ and $x = (x_1\,,x_2)$ to be

$$x_1 = \langle z|\hat{x}_1|z\rangle = \frac{L}{\sqrt{(M-1/2)\theta}}z_1\,, \qquad x_2 = \langle z|\hat{x}_2|z\rangle = \frac{2\sqrt{(M-1/2)\theta}}{L}z_2\,, \tag{4.4}$$

or

$$|z|^2 = z_1^2 + z_2^2 = \left(M - \frac{1}{2}\right)\theta\frac{x_1^2}{L^2} + \frac{L^2}{4\theta(M-1/2)}x_2^2\,. \tag{4.5}$$

Substituting (4.5) into (4.2) we define the l.h.s. of the resulting expression by

$$\mathcal{Q}_{M-1}(x_1\,,x_2) \equiv \mathcal{P}_{M-1}(z,\bar{z})\,. \tag{4.6}$$

From eqn. (65) we see $\mathcal{P}_{M-1}$ has maxima [2] at the value $z = z_0$ given by $|z_0|^2 = \theta(M-1)$. This implies that the function $\mathcal{Q}_{M-1}(x_1\,,x_2)$ has maxima on an ellipse given by

$$\frac{x_1^2}{\left(\frac{M-1}{M-1/2}\right)L^2} + \frac{x_2^2}{\frac{4\theta^2}{L^2}(M-1)(M-1/2)} = 1\,, \tag{4.7}$$

the axes of the ellipse being given by $\sqrt{\frac{M-1}{M-1/2}}L$ and $\frac{2}{L}\theta\sqrt{(M-1)(M-1/2)}$ respectively. Now holding θ constant and taking M large we see from figure 1 that the maxima of

[2] We thank Fedele Lizzi for his private communications to us regarding the nature of the maxima. His comments together with the discussion in the paper by Lizzi, Vitale and Zampini [8] have led us to a better understanding of this issue.

$\mathcal{Q}_{M-1}(x_1\,,x_2)$ have the geometry of an ellipse which is extended along the x_2 direction. For large M and fixed θ and L this conclusion is also seen from the ratio of semimajor axis of the ellipse to its semiminor axis, which is approximately $\frac{2}{L}\,\theta\, M : L$. From this reasoning and from figure 1 and figure 2 we conclude that the extended elliptical geometry in the limit $M \to \infty$ converges to an infinite strip along the x_2 axis with peaks at $x_1 = \pm L$. This proves our assertion in section 3.2 about the behavior of diagonal coherent state matrix elements of operators on $\mathcal{H}_M$ for large M.

In order to get a measure of the sharpness of the maxima of $\mathcal{Q}_{M-1}$, we now compute the width of the function $\mathcal{Q}_{M-1}$ at $(x_1 = \sqrt{\frac{M-1}{M-1/2}}L\,, x_2 = 0)$, the computation for $(x_1 = -\sqrt{\frac{M-1}{M-1/2}}L\,, x_2 = 0)$ being similar. It may also be noted that although we do the computation at $x_2 = 0$ our result is valid for all $|x_2|$ independent of M in the large M limit, where the geometry is that of an infinite strip. Let the width of the maxima of the function $\mathcal{Q}_{M-1}$ by denoted by Δx_1. We estimate Δx_1 by imposing the requirement that $\mathcal{Q}_{M-1}(x_1 + \frac{1}{2}\Delta x_1\,, x_2 = 0) = \frac{1}{e}\mathcal{Q}_{M-1}(x_1\,, x_2 = 0)$. For large values of M, the width Δx_1 can be obtained (using Maple) as

$$\Delta x_1|_{M\to\infty} \to \sqrt{\frac{2}{M}}\,L + \mathcal{O}\left(\frac{1}{M}\right). \tag{4.8}$$

Thus the width Δx_1 gets narrower as M gets larger. It is however, important to remark the following: The value of the maxima of $\mathcal{Q}_{M-1}$ for large M is proportional to $\frac{1}{\theta\sqrt{M-1}}$. This implies that for fixed θ, the height of the peaks decreases as M gets larger. Nevertheless, the normalization integral $\int \mathcal{Q}_{M-1}(x_1\,,x_2)d^2x = 1$ is preserved. A rough estimate of the volume under the graph of $\mathcal{Q}_{M-1}$ would be enough to see that this is the case. For large values of M, the width $\Delta x_1 \approx 1/\sqrt{M}$, the height $h \approx 1/\sqrt{M}$ and x_2 extends to order M. Thus for large values of M, the volume under the graph is independent of M.

It is natural to take the thickness $\Delta\ell$ of the 'fuzzy edge"'s in our matrix model to be given by the characteristic width Δx_1 of $\mathcal{Q}_{M-1}$. As $M \to \infty$ we have $\Delta x_1 = L/\sqrt{M} \to 0$, and consequently we get sharp boundaries (i.e. thin edges) at $\pm L$. Thus this limit defines the non-commutative CS theory on the infinite strip.

We end this section by estimating the spacing between the edge states of our model. First recall that for a given $\eta \in \mathbb{Z}^+$, the number of edge states in our matrix model is $\sqrt{(\eta+2)^2 - 1} \approx \eta + 2$. Let $\alpha(M)$ be the spacing of these edge states. As argued before, the thickness $\Delta\ell$ of the "fuzzy edge" is given by Δx_1. We therefore have

$$\alpha(M)(\eta + 2) = \Delta x_1\,. \tag{4.9}$$

However, we note that by construction of our model we have $0 \leq \eta < M$. This enables us to extract both an upper bound and a lower bound for the spacing of the edge states $\alpha(M)$. Thus in the large M limit, using (71) and (72), we find that

$$2\,\alpha(M) \leq \Delta\ell = \sqrt{\frac{2}{M}}\,L < M\alpha(M) \qquad \Rightarrow \qquad \frac{\sqrt{2}L}{M^{\frac{3}{2}}} < \alpha(M) \leq \frac{L}{\sqrt{2}\,M^{1/2}}\,. \tag{4.10}$$

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5. Commutative limit

In this section, we would like to comment on the commutative limit of our matrix model defined by $M \to \infty$ and $\theta \to 0$. First, we define $F(z\,,\bar{z}) := \langle z|\hat{F}|z\rangle$ for an arbitrary operator $\hat{F}$. The standard star product is then given by

$$F \star G(z) = \langle z|\hat{F}\hat{G}|z\rangle\,. \tag{5.1}$$

Making use of the formula

$$\int d^2z\, F(z\,,\bar{z}) = 2\pi\,\theta\,\mathrm{Tr}\;\hat{F}\,, \tag{5.2}$$

we can express (3.43) in the diagonal coherent state representation as

$$\{q(\Lambda_1)\,, q(\Lambda_2)\} = -q([\Lambda_1\,,\Lambda_2]_\star) - \frac{\kappa}{2\pi}\varepsilon_{ij}\int d^2z\,\mathbf{1}_M \star (\partial_i\Lambda_1) \star \mathbf{1}_M \star (\partial_j\Lambda_2)\,. \tag{5.3}$$

The first term in r.h.s. of the above expression vanishes as $\theta \to 0$ since the $\star$-product becomes the ordinary product and the commutator of functions Λ_i under ordinary product is zero. Also in this limit, we see from (3.12) that $\hat{\mathbf{1}}_M \to \mathbf{1}$. Using these in (5.3) we recover the standard $U(1)$ Kac-Moody algebra of the edge observables of commutative CS theory:

$$\{q(\Lambda_1)\,, q(\Lambda_2)\} = -\frac{\kappa}{4\pi}\,\varepsilon_{ij}\int d^2x\,\widetilde{(\partial_i\Lambda_1)}\,\widetilde{(\partial_j\Lambda_2)}\,. \tag{5.4}$$

In getting this final result we have made use of the identity [13]

$$\int d^2z\, F(z,\bar{z}) = \frac{1}{2}\int d^2x\,\widetilde{F}(x_1,x_2)\,, \tag{5.5}$$

where $\widetilde{F}(x_1,x_2)$ denotes the Moyal representation of the operator $\hat{F}$, and the fact that the $\star_M$ is removed under integration over the Moyal plane.

Finally it may be noted that the parameter θ may tend to zero in different fashions resulting to different geometries in the commutative limit. For example, taking $\theta \approx 1/\sqrt{M}$ still gives a strip geometry in the large M limit. In this case the height of the maxima of the function $\mathcal{Q}_{M-1}(x_1\,,x_2)$ at $x_1 = \pm L$ is constant, x_2 extends to $\pm\sqrt{M}$ and Δx_1 varies as $1/\sqrt{M}$. To be more precise for all $\theta M \to \infty$ as $M \to \infty$ results in a strip geometry. On the other hand, as discussed by Lizzi, Vitale and Zampini in [8], the $\theta \to 0$ limit keeping the product θM fixed gives the geometry of a disc.

6. Concluding remarks

In this work we have formulated the Chern-Simons theory on an infinite strip on the Moyal plane. Our formulation involved the construction of a new matrix CS model whose features are associated with the underlying Moyal plane. Performing canonical analysis revealed that this matrix model has "fuzzy" edges whose thickness $\Delta\ell$, we found out to be inversely proportional to $\sqrt{M}$ in the large M limit. Thus in this limit our matrix model approximated the non-commutative CS theory on the infinite strip. Our results show that the edge observables on the boundaries of the non-commutative infinite strip are given by a finite-dimensional Lie algebra with properties similar to that of a non-abelian Kac-Moody algebra. Our findings generalize the well-known results of the usual CS theory to non-commutative manifolds with boundaries, thereby opening new possibilities for the treatment of non-commutative Quantum Hall Effect and for other applications in such domains.

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